Pseudo-Riemannian Geometry and the Gauss-Bonnet Formula*)

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1. INTRODUCTION

Recent investigations in the theory of general relativity have aroused an interest in lorentzian and, more generally, in pseudo-riemannian manifolds. The latter are manifolds with non-degenerate quadratic differential forms defined on them. To a compact oriented pseudo-riemannian manifold the integral in the Gauss-Bonnet formula can be generalized, and it is natural to ask whether its value is still equal to the Euler-Poincaré characteristic of the manifold. The usual proof (see [1], [3]) cannot be extended immediately. It is the main purpose of this paper to prove that the formula, with a suitable modifiction, remains true.

We will develop pseudo-riemannian geometry in more detail than would be necessary for the treatment of the Gauss-Bonnet formula. This is partly motivated by the increasing interest on the subject, but a more pertinent reason is the fact that in the study of pseudo-riemannian manifolds the geometry of pseudo-riemannian vector bundles plays a very essential rôle.

2. PSEUDO-RIEMANNIAN VECTOR BUNDLES

By a manifold we will always mean a connected, paracompact, C^∞-differentiable manifold. Moreover, all our functions and mappings will be understood to be C^∞.

Let M be a manifold of dimension m. Let $\psi: B \longrightarrow M$ be a bundle of real vector spaces of dimension r over M. This means that there is an open covering $\{U, V, W, \ldots\}$ of M such that $\psi^{-1}(U)$ can be coordinatized, relative to U, by the coordinates (x, y_U), $x \in U$, $y_U \in Y$, where Y is a typical fiber, which is a real vector space of dimension r. Moreover, if $x \in U \cap V$ and if (x, y_V), $y_V \in Y$, are the local coordinates of $\psi^{-1}(x)$ relative to V, then $y_U = \gamma_{UV}\, y_V$, where γ_{UV} is a mapping: $U \cap V \longrightarrow GL(r; R)$. The mappings γ_{UV}, to be called the transition functions of the bundle relative to the covering $\{U, V, \ldots\}$, satisfy the conditions

$$\gamma_{UU} = \text{identity},$$

$$(1) \qquad \gamma_{UV}^{-1} = \gamma_{VU},$$

$$\gamma_{UV}\,\gamma_{VW}\,\gamma_{WU} = \text{identity in } U \cap V \cap W \neq \varnothing.$$

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We will frequently consider Y to be the vector space of one-columned vectors and GL(r;R) to be the group of all non-singular (r $\times$ r)-matrices; the action of GL(r; R) on Y will then be given by matrix multiplication.

The vector bundle B is called pseudo-riemannian, if there exists a non-degenerate symmetric bilinear function H in each fiber $\psi^{-1}(x)$, which varies in a C^∞-way in x. Since M is connected, the signature (p, q) of H, $p + q = r$, is constant. (If H is considered as a quadratic form in $\psi^{-1}(x)$ and reduced to square terms by a choice of basis in $\psi^{-1}(x)$ then p is the number of positive squares and q is the number of negative squares). A pseudo-riemannian bundle is called riemannian if $q = 0$ and lorentzian if $q = 1$.

A manifold with a pseudo-riemannian (respectively riemannian or lorentzian) structure in its tangent bundle is called pseudo-riemannian (respectively riemannian or lorentzian).

As mentioned above, we will consider the transition functions γ_{UV} as non-singular (r $\times$ r)-matrices defined in $U \cap V \neq \emptyset$ for any two neighborhoods U, V of our covering. Relative to the same covering, an affine connection is given by an (r $\times$ r)-matrix θ_U of linear differential forms in each U such that, in $U \cap V \neq \emptyset$, we have

$$d\,\gamma_{UV} + \theta_U\,\gamma_{UV} = \gamma_{UV}\,\theta_V. \tag{2}$$

It is easy to verify that this relation is compatible, that is, in $U \cap V \cap W \neq \emptyset$, two of these relations imply the third one as a consequence.

From (2) it follows that

$$d\,y_U + \theta_U\,y_U = \gamma_{UV}\,(d\,y_V + \theta_V\,y_V). \tag{3}$$

The vanishing of this common expression is therefore independent of the coordinate neighborhood. If a vector field is defined on a curve C in M (i.e., if a cross-section of the vector bundle B over C is given), it is said to be parallel along C if (3) vanishes.

An affine connection on a pseudo-riemannian vector bundle is called admissible, if the bilinear function H(y, z), $y, z \in \psi^{-1}(x)$, remains unchanged when the vectors y, z are displaced parallelly along curves. We will call H(y, z) the scalar product of y, z and we will find the condition for an affine connection to be admissible. Relative to a coordinate neighborhood U, let y_U, z_U be the column vectors which are the coordinates of y, z. Then the function H(y, z) can ve written in the matrix form

$$H(y, z) = {}^t y_U\,H_U\,z_U, \tag{4}$$

where ${}^t y_U$ denotes the transpose of y_U and H_U is a non-singular symmetric matrix in U. The transformation law for H_U is obviously

$$H_V = {}^t\gamma_{UV}\,H_U\,\gamma_{UV}. \tag{5}$$

Under parallel displacement of the vectors y, z we have

$$d({}^t y_U\,H_U\,z_U) = {}^t y_U\,(-\,{}^t\theta_U\,H_U + d\,H_U - H_U\,\theta_U)\,z_U.$$

Since this is to be zero for arbitrary y, z, the condition for the affine connection to be admissible is

$$d\,H_U - H_U\,\theta_U - {}^t\theta_U\,H_U = 0. \tag{6}$$

It is important now to derive the relations which will follow by exterior differentiation of (2) and (6). Taking the exterior derivative of (2), we get

(7) $$\Theta_V = \gamma_{UV}^{-1}\,\Theta_U\,\gamma_{UV},$$

where

(8) $$\Theta_U = d\,\theta_U + \theta_U \wedge \theta_V.$$

The last product is the usual product of matrices, with the additional convention that multiplication of exterior differential forms is in the sense of the exterior product. The matrices Θ_U of exterior differential forms of degree two, with the transformation law (7), define the curvature of the affine connection; Θ_U will be called the curvature form (relative to U).

Similarly, exterior differentiation of (6) gives

(9) $$H_U\,\Theta_U + {}^t\Theta_U\,H_U = 0,$$

so that the matrix $H_U\,\Theta_U$ is anti-symmetric. Moreover, it satisfies the transformation law

(10) $$H_V\,\Theta_V = {}^t\gamma_{UV}\,H_U\,\Theta_U\,\gamma_{UV}.$$

Comparison of (5) and (10) leads to a globally defined form on M in case r is even and the bundle B is oriented. The latter means that the group of the bundle is, instead of GL(r, R), its onnected component of the identity or, what is the same, that the matrices γ_{UV} have positive determinants. From (5) we get

$$\det H_V = (\det \gamma_{UV})^2\,(\det H_U).$$

Under the assumption that the bundle is oriented, this implies

(11) $$|\det H_V|^{\frac{1}{2}} = (\det \gamma_{UV})\,|\det H_U|^{\frac{1}{2}},$$

where the square roots are taken to be positive.

We put

(12) $$H_U\,\Theta_U = (\Theta^{\alpha\beta},\ U),\quad \gamma_{UV} = (\gamma^{\beta}_{\alpha,\,UV}),\ 1 \leqq \alpha, \beta \leqq r.$$

Consider the expression

$$\Sigma\,\epsilon_{\alpha_1\ldots\alpha_r}\,\Theta^{\alpha_1\alpha_2}{}_{,V}\ldots\Theta^{\alpha_{r-1}\alpha_r}{}_{,V},$$

where $\epsilon_{\alpha_1\ldots\alpha_r}$ is $+1$ or -1 according as $\alpha_1,\ldots,\alpha_r$ are an even or odd permutation of $1,\ldots,r$, and is otherwise zero, and where the summation is extended over all $\alpha_1,\ldots,\alpha_r$ from 1 to r. By (10) this expression is equal to the corresponding expression with the subscript U, multiplied by $\det \gamma_{UV}$. It follows that the form

(13) $$\Delta = \frac{(-1)^{\frac{r}{2}}}{2^r\,\pi^{\frac{r}{2}}\left(\frac{r}{2}\right)!\,|\det H_U|^{\frac{1}{2}}}\,\Sigma\,\epsilon_{\alpha_1\ldots\alpha_r}\,\Theta^{\alpha_1\alpha_2}{}_{,U}\ldots\Theta^{\alpha_{r-1}\alpha_r}{}_{,U}$$

is independent of the subscript U and is globally defined in M. It is not hard to verify that $d\Delta = 0$. Hence Δ defines, in the sense of the Rham's theorem, an element of the cohomology group $H^{2r}(M:R)$ of M with real coefficients. We will show that this cohomology class is the Euler characteristic class of the vector bundle B. For riemannian bundles this result is known as the Gauss-Bonnet formula.

If a scalar product H defines a pseudo-riemannian structure of signature (p, q) on a vector bundle B, its negative $-H$ defines a pseudo-riemannian structure of signature (q, p). An affine connection admissible to the one is admissible to the other.

From two vector bundles over the same manifold M their Whitney sum can be constructed. If the bundles are pseudo-riemannian, a pseudo-riemannian structure can be defined on their Whitney sum in an obvious way. The same can be said about admissible affine connections.

3. THE WEIL HOMOMORPHISM

The notion of a pseudo-riemannian vector bundle and its admissible affine connection is a special case of the notion of a connection in a principal fiber bundle with a Lie group. We will establish this relationship and apply a theorem of Weil ([2], [4]) to the effect that the cohomology class determined by Δ is independent of the choice of the connection. For this purpose we recall that a frame of the bundle B is an ordered set of r linearly independent vectors with the same origin x (i.e., in the same fiber $\psi^{-1}(x)$). Relative to a coordinate neighborhood U, a frame can be defined analytically by a nonsingular $(r \times r)$-matrix s_U, whose columns are the components of the r vectors. All the frames of B form the frame bundle, which we will denote by B_F. The projection we will denote by $\psi_F: B_F \longrightarrow M$. The frame bundle has the local coordinates (x, s_U), with the transformation law $s_U = \gamma_{UV} s_V$ valid in $U \cap V$.

In $\psi_F^{-1}(U)$ we introduce the matrix of linear differential forms

$$\varphi = s_U^{-1} d s_U + s_U^{-1} \theta_U s_U . \tag{14}$$

Equation (2) is then equivalent to the statement that this expression is equal, in $\psi_F^{-1}(U \cap V)$, to the corresponding expression with the subscripts U replaced everywhere by V. In other words, φ is globally defined in B_F. If we put

$$\Phi = d\varphi + \varphi \wedge \varphi , \tag{15}$$

we find

$$\Phi = s_U^{-1} \Theta_U s_U . \tag{16}$$

Thus the matrix Φ, globally defined in B_F, essentially gives the curvature form of the connection.

Similarly, we introduce in B_F the matrix

$$F = {}^t s_U H_U s_U = {}^t s_V H_V s_V . \tag{17}$$

This matrix has a simple geometrical meaning: The element in its αth row and βth column is the scalar product of the αth and the βth vectors of the frame. The admissibility of the affine connection can be expressed by the condition

$$dF = F\varphi + {}^t\varphi F . \tag{18}$$

Exterior differentiation of (18) gives

$$F\Phi + {}^t\Phi F = 0 \tag{19}$$

If the bundle is oriented and r is even, the expression in (13) can be written

$$(20)\qquad \psi_F^{*}\,\Delta = \frac{(-1)^{\frac{r}{2}}}{2^r\,\pi^{\frac{r}{2}}\left(\frac{r}{2}\right)!\,|\det F|^{\frac{1}{2}}}\sum \epsilon_{\alpha_1\ldots\alpha_r}\,\Phi^{\alpha_1\alpha_2}\ldots\Phi^{\alpha_{1-r}\alpha_r}, \qquad F\,\Phi = (\Phi^{\alpha\beta}).$$

The advantage for the consideration of the frame bundle consists in the fact that the quantities are globally defined in it instead of being locally defined in the base manifold.

We now restrict ourselves to frames for which the matrix F is diagonal with the diagonal elements $(\underbrace{1,\ldots,1}_{p},\underbrace{-1,\ldots,-1}_{q})$. This matrix we will denote by $F_o(p, q)$ or simply F_o. The frames in question are then characterized analytically by the condition

$$(21)\qquad F_o = {}^t s_U\, H_U\, s_U\,.$$

Their totality constitutes a submanifold B_{FO} of B_F. By reducing the coordinate neighborhoods when necessary, we choose in each U a matrix $s_U(x)$, $x \in U$, of positive determinant and satisfying the condition (21). In the vector bundle B we can use new local coordinates defined by

$$(22)\qquad y'_U = s_U^{-1}\, y_U\,.$$

The new transition functions will then be

$$(23)\qquad \gamma'_{UV} = s_U^{-1}\,\gamma_{UV}\, s_V\,,$$

and they satisfy the condition

$$(24)\qquad {}^t\gamma'_{UV}\, F_o\, \gamma'_{UV} = F_o.$$

We wish also to note that the scalar product in terms of the new local coordinates is given by

$$(25)\qquad H(y, z) = {}^t y'_U\, F_o\, z'_U\,.$$

In general, we will denote by O(p,q) the group of all $(r \times r)$-matrices T satisfying the condition

$$(26)\qquad {}^tT\, F_o\, T = F_o,$$

and by SO(p, q) the subgroup of O(p, q) whose matrices have positive determinant. We set $O(r, 0) = O(r)$, $SO(r, 0) = SO(r)$, which are respectively the orthogonal group and the properly orthogonal group in r variables.

Since $\gamma'_{UV} \in O(p, q)$, the pseudo-riemannian structure of B gives a reduction of the structure group to O(p, q). Moreover, if B is oriented, its structure group is reduced to SO(p, q). Conversely, a reduction of the structure group of B to O(p, q) (respectively to SO(p, q)) implies a pseudo-riemannian (respectively an oriented pseudo-riemannian) structure on B. For the scalar product defined by (25) is then independent of the choice of the coordinate neighborhood U.

Suppose that our vector bundle B has an oriented pseudo-riemannian structure. Its structure group is then reduced to SO(p,q), and it has as its principal fiber bundle the sub-manifold B_{FSO} of B_F, which consists of all frames satisfying (21) and the additional condition that $\det s_U > 0$. Given an admissible affine connection, the restriction of φ

to B_{FSO} gives a connection form in the sense of a connection in a principal fiber bundle, while the restriction of Φ gives its curvature form. Equations (18) and (19) can then be written

$$F_0 \varphi + {}^t\varphi F_0 = 0, \qquad (27)$$
$$F_0 \Phi + {}^t\Phi F_0 = 0.$$

Weil's homomorphism gives the following result:

For an oriented pseudo-riemannian bundle of vector spaces of even dimension, the cohomology class with real coefficients in M determined by the form Δ is independent of the choice of the admissible affine connection.

4. THE GAUSS-BONNET THEOREM

We will prove the following theorem:

Let B be an oriented pseudo-riemannian bundle of vector spaces of even dimension over a compact manifold M. The cohomology class determined by the form Δ is its Euler characteristic class.

For a riemannian bundle this is the classical Gauss-Bonnet theorem. (The theorem has so far only been formulated for the tangent bundle, but the proof extends in a straightforward way).

Consider therefore the general case that the pseudo-riemannian structure H is of arbitrary signature (p,q). We impose in addition a riemannian structure G on the bundle, so that $G(y,z)$, y, $z \in \psi^{-1}(x)$, is a positive definite symmetric bilinear function, which varies in a C^∞-way with x. For fixed $x \in M$ and fixed $y \in \psi^{-1}(x)$, the eigenvalues of H relative to G are the values λ such that

$$H(y,z) = \lambda\, G(y,z)$$

holds identically in z. The corresponding non-zero y which satisfies this equation is called an eigenvector. It is well-known that there are r real eigenvalues, of which p are positive and q are negative. Let $\psi_1^{-1}(x)$ (respectively $\psi_2^{-1}(x)$) be the subspace of $\psi^{-1}(x)$, which is spanned by the eigenvectors with positive (respectively negative) eigenvalues. Then $\psi^{-1}(x)$ is a direct sum of $\psi_1^{-1}(x)$ and $\psi_2^{-1}(x)$, so that any $y \in \psi^{-1}(x)$ can be written in a unique way as a sum:

$$y = P_1 y + P_2 y, \quad P_i y \in \psi_i^{-1}(x), i = 1,2. \qquad (28)$$

We define the symmetric bilinear functions

$$H_1(y,z) = H(P_1 y, P_1 z), \qquad (29)$$
$$H_2(y,z) = -H(P_2 y, P_2 z).$$

Then H_i is positive definite when restricted to $\psi_i^{-1}(x)$, $i = 1,2$, and we have

$$H(y,z) = H_1(y,z) - H_2(y,z). \qquad (30)$$

Moreover, the function

$$K(y,z) = H_1(u,z) + H_2(y,z) \qquad (31)$$

defines a riemannian structure on the bundle B.

We introduce the bundles

(32) $$B_i = \bigcup_{x \in M} \psi_i^{-1}(x), \quad i = 1, 2,$$

whose projections $\psi_i : B_i \longrightarrow M$ are defined by $\psi_i\,(\psi_i^{-1}(x)) = x$. Then the given bundle B is a Whitney sum of B_1 and B_2. The function H_i defines a riemannian structure on B_i, from which the given pseudo-riemannian structure H and a new riemannian strucuture K on B are defined by (30) and (31). Take in B_i an affine connection admissible to H_i. Their direct sum is an affine connection in B, which is admissible to both H and K. Let Δ_H and Δ_K be the form Δ in (20) constructed with respect to the pseudo-riemannian strucutures H and K respectively.

To study these two expressions remember that they are constructed from the matrix $F\Phi$, where Φ is the curvature form of the admissible affine connection and the element in the αth row and βth column of F is the scalar product of the αth and βth vectors of the frame. Since the form Δ is in the base manifold M, we can restrict ourselves to frames at x, whose first p vectors are in $\psi_1^{-1}(x)$ and whose last q vectors are in $\psi_2^{-1}(x)$. Then the matrices F relative to H and K are respectively of the forms

$$F_H = \begin{pmatrix} F_1 & 0 \\ 0 & -F_2 \end{pmatrix}, \qquad F_K = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix},$$

where F_1, F_2 are positive definite symmetric matrices of orders p,q respectively. Moreover, the curvature matrix of the affine connection when restricted to our choice of frames is of the form

$$\Phi = \begin{pmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{pmatrix},$$

where Φ_1, Φ_2 are matrices of exterior quadratic differential forms of orders p, q respectively. From these expressions we immediately conclude that

$$\Delta_H = \Delta_K = 0, \text{ if p or q is odd,}$$

and that

$$\Delta_H = \Delta_K, \text{ if p and q are even.}$$

In both cases we see that the theorem is true in case the affine connection is the one chosen above. Our theorem then follows as a consequence of the theorem at the end of § 3.

Our discussion gives also a new proof, and slight generalization, of a theorem of H. SAMELSON and T. J. WILLMORE [5], [6].

If an oriented bundle over a compact manifold has a field of subspaces of odd dimension, its Euler class with real coefficients is zero.

5. PSEUDO-RIEMANNIAN MANIFOLDS

Certain special features arise, when the vector bundle is the tangent bundle of the manifold M, and we will discuss them. In this case the transition function γ_{UV} is the functional matrix of the local coordinates u^i in U with respect to the local coordinates v^k in V, $1 \leqq i, k \leqq m$. We introduce the one-rowed matrices

(33) $$du = (du^1, \ldots, du^m), \quad dv = (dv^1, \ldots, dv^m).$$

Then

$$du\ s_U = dv\ s_V \quad \text{in } U \cap V \tag{34}$$

and this common expression defines, globally in the frame bundle B_F, a one-rowed matrix of linear differential forms, which we will denote by τ. If there is an affine connection in the bundle, we will have, in $\psi_F^{-1}(U)$,

$$d\tau = -\tau \wedge \varphi + du \wedge \Theta_U\, S_U .$$

The affine connection is said to be without torsion, if the following condition holds:

$$d\tau = -\tau \wedge \varphi . \tag{35}$$

If we put

$$\theta_U = \left(\sum_k \Gamma^j_{ik}\, du^k\right), \quad 1 \leqq i,\ j,\ k \leqq m,$$

then

$$du \wedge \theta_U = \left(\sum_{i,k} \Gamma^j_{ik}\, du^i \wedge du^k\right),$$

and the absence of torsion of the affine connection is equivalent to the analytical condition that Γ^j_{ik} is symmetric in i, k.

The classical argument on the existence and uniqueness of the Levi-Civita connection of a riemannian metric, which we will not respeat here, shows that there is exactly one matrix Θ_U which satisfies the condition (6) and the condition: $du \wedge \Theta_U = 0$. In other words, we have the following result: *Every pseudo-riemannian structure on a manifold has exactly one admissible affine connection without torsion.*

Another special feature for the pseudo-riemannian structure in a tangent bundle is the possibility of introducing the sectional curvature to every two-dimensional subspace of the tangent space. Without introducing more analytical apparatus, we will show how the notion of sectional curvature in the cotangent space can be defined. From here till the end of the paper we will suppose that all small Latin indices have the range from 1 to m. We will restrict ourselves to a coordinate neighborhood U and will compare the quantities defined in it with quantities in $\psi_F^{-1}(U)$. We put

$$s_U = (s^j_i), \quad H_U = (h^{ij}), \quad F = (f^{ij}), \tag{36}$$

so that, by (17), we have

$$f^{ij} = \sum_{k,l} s^i_k\, s^j_l\, h^{kl} . \tag{37}$$

Let

$$\tau = (\tau^1, \ldots, \tau^m) . \tag{38}$$

Then we have, by (34),

$$\tau^i = \sum_j du^j\, s^i_j \tag{39}$$

A bivector in the cotangent space can be written

$$\xi = \sum_{i,j} p_{ij}\, du^i \wedge du^j = \sum_{i,j} q_{ij}\, \tau^i \wedge \tau^j, \quad p_{ij} + p_{ji} = q_{ij} + q_{ji} = 0, \tag{40}$$

where

(41) $$p_{ij} = \sum_{k,l} q_{kl}\, s_i^k\, s_j^l .$$

Introduce the quantities

(42) $$p^{ij} = \sum_{k,l} h^{ik}\, h^{jl}\, p_{kl}, \qquad q^{ij} = \sum_{k,l} f^{ik}\, f^{jl}\, q_{kl} .$$

Then we have

(43) $$q^{ij} = \sum_{a,b} s_a^i\, s_b^j\, p^{ab},$$

and

(44) $$\frac{1}{2}\sum_{i,j} p_{ij}\, p^{ij} = \frac{1}{2}\sum_{i,j} q_{ij}\, q^{ij} .$$

This common expression is, from the right-hand side, independent of U and, from the left-hand side, independent of the frame. Its value is therefore a quantity associated to the bivector ξ; it is the square of the measure of ξ.

Similarly, we write

(45) $$H_U\, \Theta_U = (\Theta^{ij}), \quad F\, \Phi = (\Phi^{ij}),$$

where

(46) $$\Theta^{ij} = \frac{1}{2}\sum_{k,l} R^{ij}_{kl}\, du^k \wedge du^l, \qquad \Phi^{ij} = \frac{1}{2}\sum_{k,l} S^{ij}_{kl}\, \tau^k \wedge \tau^l,$$

and

(47) $$R^{ij}_{kl} = -R^{ij}_{lk} = -R^{ji}_{kl}, \quad S^{ij}_{kl} = -S^{ij}_{lk} = -S^{ji}_{kl} .$$

Then we have, from (16), (17), and (39),

(48) $$\sum_{i,\dots,l} R^{ij}_{kl}\, p_{ij}\, p^{kl} = \sum_{i,\dots,l} S^{ij}_{kl}\, q_{ij}\, q^{kl} .$$

This common value is therefore a quantity associated to ξ by the curvature of the connection.

The quotient

(49) $$K = -\sum_{i,\dots,l} R^{ij}_{kl}\, p_{ij}\, p^{kl} \Big/ 2\sum_{i,j} p_{ij}\, p^{ij}$$

depends only on the two-dimensional subspace of the cotangent space determined by ξ. It is called its sectional curvature.

Added December 11, 1962. After this paper has gone to press, there appeared a note of ANDRÁ AVEZ: *Formule de Gauss-Bonnet-Chern en métrique de signature quelconque*, Comptes Rendus de l'Acad. des Sciénces (Paris), 255, 2049–2051 (1962). In this note Avez sketched a proof of our main result. His proof has some contact with ours, but is somewhat different".

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